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CONCERNING SERIES OF ANALYTIC FUNCTIONS

By M. B. Porter

The purpose of this note is to establish the following theorem:

If a series of functions, $f_1(z), f_2(z), \cdots$, *analytic in a region* γ *is such that throughout* γ

$$\text{(I)} \qquad |S_n(z)| = |f_1(z) + \cdots + f_n(z)| < G, \qquad (n = 1, 2, \cdots)^*$$

and

(II) $\lim\limits_{n=\infty} S_n(z)$ *exists for some infinite point-set* $[Z_i]$, *one, at least, of whose limiting points lies inside of* γ, *then* $S_n(z)$ *converges uniformly throughout any region* γ' *lying inside of* γ *and represents an analytic function there.*

To prove this let us consider the point-set in the complex plane $w = S_n(z_0)$, $n = 1, 2, \cdots$, where z_0 is any point of γ'. By (I) these points will cluster about a point-set $P(z_0)$. Let $P_0(z_0)$ be one of these limiting points; then we can pick out of the set $S_n(z_0)$ a partial sequence $S(n, 1, z_0)$ such that

$$|P_0(z_0) - S(n, 1, z_0)| < \epsilon, \qquad n > N_1.$$

Let us now consider any other point z_1 of γ'; in the same way we can pick out of the sequence $S(n, 1, z_1)$ a partial sequence $S(n, 2, z)$ such that we have simultaneously

$$\begin{aligned} |P_0(z_0) - S(n, 2, z_0)| &< \epsilon, \\ |P_1(z_1) - S(n, 2, z_1)| &< \epsilon, \end{aligned} \qquad n > N_2,$$

where $P_1(z_1)$ denotes one of the limiting points of $S(n, 1, z_1)$.

Let us now imagine m points $z_0, z_1, z_2, \cdots z_{m-1}$ (denote the set by $[m]$) so distributed over the region γ' that every point of γ' is at a distance less than δ_0 from at least one of the points of the set $[m]$.

From the foregoing we see that we can pick out of $S(n, 1, z)$ a partial sequence which may be denoted by $S(n, m, z)$ such that

$$|P_i(z_i) - S(n, m, z_i)| < \epsilon, \qquad \begin{cases} n > N_m, \\ i = 0, 1, 2, \cdots m - 1. \end{cases}$$

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* This condition can be replaced by the equivalent one $|f_1(z) + f_2(z) + \ldots + f_n(z)| < G$ $(n = 1, 2, \ldots)$ for all the points of the boundary of γ (supposed rectifiable).

Suppose now that we increase m by *adjoining* new points in such a way that δ_0 takes on a set of values $\delta_1, \delta_2, \cdots \delta_i, \cdots$ approaching zero; we shall get a set of sequences $S(n, m, z)$, $S(n, m+1, z)$, $\cdots S(n, m+p, z), \cdots$, where every sequence is a *partial* sequence of all those that precede it and is such that

$$1) \qquad |S(n, m'+p, z) - S(n, m', z)| < 2\,\epsilon, \qquad \begin{cases} n > N_{m'}, \\ p = 1, 2, \cdots, \end{cases}$$

for all points of the set $[m']$.

For any pair of points $(z, z+\delta)$ of the region γ' we have by Cauchy's Integral formula,

$$2) \qquad |S_n(z) - S_n(z+\delta)| < \frac{1}{2\pi}\int \frac{G\delta\,|dt|}{|z-t|\,|z+\delta-t|} < M\delta, \qquad (M \textit{ constant}),$$

where the integral is taken around some contour containing γ' inside of it. From 2) we have

$$3) \qquad |S(n, m', z) - S(n, m', z_j)| < M\delta_i,$$

where z_j denotes any point of the set $[m']$ such that $|z - z_j| \leqq \delta_i$; also,

$$4) \qquad |S(n, m'+p, z) - S(n, m'+p, z_j)| < M\delta_i;$$

and from 1), 3), and 4),

$$5) \qquad |S(n, m'+p, z) - S(n, m', z)| < 2\,(\epsilon + M\delta_i), \qquad n > N_{m'},$$

for *all* the points of γ' at once.

Now 5) shows that $S(n, m, z)$ converges *uniformly** to its limit throughout γ', and since it is always analytic, its limit, which we shall call $\Sigma_1(z)$, is analytic in this region.

If now we suppose that $f_1(z) + f_2(z) + f_3(z) + \cdots$ converges uniformly throughout some two-dimensional region of γ, then if γ' be so chosen as to contain this region we get a well known theorem of Stieltjes.*

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* If $S_n(x_1, x_2, \ldots, x_m)$ denote an infinite set of continuous functions of the m variables x_i, *limited* in a domain D and containing a partial sequence *equably continuous* in Arzelà's sense, the foregoing considerations show that we can always pick out at least one sequence $S_{n_i}(x_1, x_2, \ldots, x_m)$ which converges *uniformly* to a limit $S(x_1, x_2, \ldots, x_m)$ which is continuous in D (Arzelà); moreover that the number of such functions $S(x)$ is at least as great as the number of limiting points of the equably continuous sequences of S_n for any (fixed) point of the domain D.

If however we postulate merely the condition II of our theorem, $\Sigma_1(z)$ will evidently take on at the points $[Z_i]$ the same values as

$$\lim_{n=\infty} [f_1(z) + f_2(z) + \cdots + f_n(z)].$$

Suppose however that instead of starting with the point $P_0(z_0)$ to construct our function $\Sigma_1(z)$ we had started with any *other* point of the set $P(z_0)$; we should then have arrived at another (and different) analytic function $\Sigma_2(z)$ which would however take on in $[Z_i]$ the same values as $\Sigma_1(z)$. Now this is a contradiction, for since $\Sigma_1(z)$ and $\Sigma_2(z)$ take on the same values in $[Z_i]$ they coincide and hence must take on the same value at Z_0 so that the point-set $P(z_0)$ consists of but *one* point and our theorem is proved.

If we remove the restriction II of our initial theorem we have seen that our original set of analytic functions defines a second set of analytic functions at least as numerous as the points of the set $P(z_0)$; denote this set of functions by $\Sigma_1(z), \Sigma_2(z), \cdots$. Several interesting questions present themselves: Under what circumstances will the set of functions $\Sigma_i(z)$ be finite in number? What is the nature of the correspondence between the set Σ_i and the points $P(z_0)$ (where z_0 denotes a non-specialized point of γ)? The same questions present themselves, of course, for a set of equably continuous functions, in any number of real variables, subject to the condition of being limited.

UNIVERSITY OF TEXAS,
AUSTIN, TEXAS.

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* *Ann. de la Faculté des Sciences de Toulouse,* vol. 8 (1894), p. J. 56. The generalization of this theorem by Professor Osgood (ANNALS OF MATHEMATICS, ser. 2, vol. 3, p. 25, 1901), as well as the theorem by the present writer (*ibid.* vol. 6, p. 45, 1904), are special cases of the theorem at the beginning of this paper.